An Analysis of Onion Options and Double-no-Touch Digitals*

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Abstract

In this article it is shown how an onion option can be decomposed into double-no-touch options, for which the Black-Scholes framework admits a solution in terms of a Fourier series. The convergence properties of the series are studied in detail. In particular, an explicit formula for the number of terms needed to achieve a desired accuracy is presented. The hedging parameters are derived followed by a discussion of the peculiar properties of vega and gamma.

1 Introduction

In this paper we are studying a product from the FX market called the *onion option* which can be decomposed into a sum of double-no-touch options. The *double-no-touch* option [Lipton, 2001] is of double-barrier knock-out type and pays a prespecified amount of money at maturity provided that neither the upper nor the lower knock-out barrier has been breached. In principle many types of barrier options are conceivable depending on whether a barrier is hit and/or whether it is hit first. In this paper, though, we solely discuss the double-barrier knock-out type which is predominant in both, the OTC and retail market. Other types of double barrier binaries are discussed in [Luo, 2001]. Having established well-posedness of the pricing problem, we derive formulae for the price of the option and the Greeks based on Fourier series techniques. While the formulae for the price of the option have been given by [Hui, 1996] and [Lipton, 2001], formulae for the Greeks have not been published to our knowledge. The convergence properties of these formulae are studied in detail including a discussion of the Gibbs phenomenon. Easy-to-use formulae are stated that determine the number of relevant terms of the Fourier series given a certain required accuracy. Also, for some properties of some of the Greeks proof is provided. Finally, we compute the implied volatilities of a set of onion options and compare these to market quotes.

2 The product

The product that we want to study in this paper is a structured derivative first issued by Warburg Dillon Read in 1998 as *Multi-Double-Lock-Out-Warrant*. It was not issued again after 1999. In early 2002, Commerzbank issued the same option under a new name: *onion option*. The underlying is

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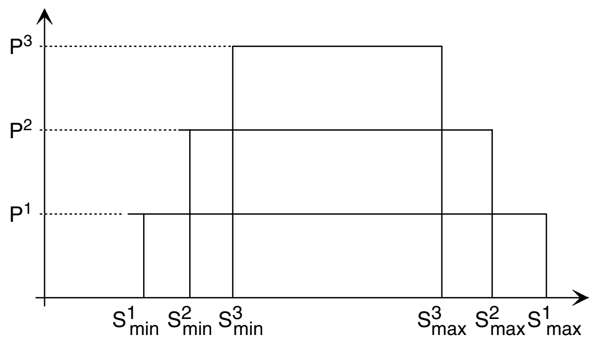

Fig. 1 Payoff of the onion option.

the USD/EUR FX rate. The payoff is defined in the following way: In case the FX rate S does not leave a corridor $S^3_{min} < S < S^3_{max}$ before maturity, the holder of the option receives a payoff of P^3. In case the FX rate leaves that corridor at least once but stays within a second outer[1] corridor $S^2_{min} < S < S^2_{max}$ the holder receives $P^2 < P^3$. In case the FX rate leaves this corridor, too, but stays within $S^1_{min} < S < S^1_{max}$ the holder gets $P^1 < P^2$. In case the FX rate does not remain within this last corridor, the option ceases to exist at the moment of breaching either S^1_{max} or S^1_{min}.
It is easy to see that — when breaching either S^1_{max} or S^1_{min} — a knock-out barrier is hit. This barrier does not pay any rebate. Therefore, the entire derivative is of the double-barrier knock-out type. Breaching either S^2_{max} or S^2_{min} has several consequences: The original contract is knocked out but not to zero. This time it is replaced by a second contract which pays P^1, if the FX rate stays between S^1_{min} and S^1_{max}. This second contract can be interpreted as a rebate. Similar arguments apply when either S^3_{max} or S^3_{min} are breached. As usual in the FX setting, the barriers are monitored continuously.

Commerzbank issued several options of this type with varying S^i_{min}, S^i_{max} and P^i; for some examples see table 1. To the knowledge of the authors only products with 3 layers have been issued so far. In the following, we show how to calculate the value of a general onion option with an arbitrary number of corridors.

3 Peeling the onion

The onion option has the following properties:

1. The onion option has n nested corridors, i.e. each of these corridors has a lower boundary S^i_{min} and an upper boundary S^i_{max} and the boundaries satisfy $0 < S^1_{min} < \ldots < S^n_{min} < S^n_{max} < \ldots < S^1_{max}$.
2. If the underlying price S does not leave the inner corridor (S^n_{min}, S^n_{max}) the option pays off P^n. If S leaves the $i+1$-th corridor but remains inside the i-th corridor the option pays off P^i. If all barriers are breached the option pays off nothing. The payoffs satisfy $0 < P^1 < \ldots < P^n$.

There is a simple trick to price this product. It can be decomposed into double-no-touch options. Let $V_{DNT}(S, t; S_{min}, S_{max}, P)$ denote the value of a double-no-touch digital defined as follows: At expiry, it pays P in case neither barrier S_{min} or S_{max} has been breached. In case any of the barriers is

TABLE 1
PRODUCT DATA OF SOME ONION OPTIONS ISSUED BY COMMERZBANK. FOR ALL THESE OPTIONS THE UNDERLYING IS THE USD/EUR FX RATE.

ISIN	Inner Corridor		Middle Corridor		Outer Corridor		Expiry
	$1/S^3_{max} - 1/S^3_{min}$ in USD	P^3 in EUR	$1/S^2_{max} - 1/S^2_{min}$ in USD	P^2 in EUR	$1/S^1_{max} - 1/S^1_{min}$ in USD	P^1 in EUR	
DE0006527922	0.825–0.925	30	0.800–0.950	20	0.775–0.975	10	8/22/2002
DE0006527930	0.825–0.925	30	0.810–0.940	20	0.800–0.950	10	8/22/2002
DE0006527948	0.840–0.910	30	0.825–0.925	20	0.810–0.940	10	8/22/2002
DE0006527955	0.850–0.900	30	0.835–0.915	20	0.820–0.930	10	8/22/2002
DE0006527963	0.800–0.950	30	0.750–1.000	20	0.700–1.050	10	3/20/2003
DE0006527971	0.800–0.950	30	0.775–0.975	20	0.750–1.000	10	3/20/2003
DE0006527989	0.825–0.925	30	0.800–0.950	20	0.775–0.975	10	3/20/2003
DE0006527997	0.825–0.925	30	0.810–0.940	20	0.795–0.955	10	3/20/2003

hit or crossed, the options ceases to exist instantaneously without paying any rebate. Then the value $V_{onion}(S,t)$ of the onion option can be written as

$$V_{onion}(S,t) = \sum_{i=1}^{n} V_{DNT}(S,t,S^i_{min},S^i_{max},P^i - P^{i-1}), \tag{1}$$

where we have introduced $P^0 = 0$. In order to prove this we will show that the right-hand side always yields the same payoff as the left-hand side: First, we note that by definition both sides are equal when the onion option only has a single corridor, i.e., $n = 1$. Assuming that Eq. (1) is correct for an onion option V^{j-1}_{onion} having only the $j-1$ outer corridors $S^1_{min} < \ldots < S^{j-1}_{min} < S^{j-1}_{max} < \ldots < S^1_{max}$ Eq. (1) can be rewritten as

$$V^j_{onion}(S,t) = V_{DNT}(S,t,S^j_{min},S^j_{max},P^j - P^{j-1}) + V^{j-1}_{onion}(S,t). \tag{2}$$

where $V^j_{onion}(S,t)$ denotes the value of an onion option with j corridors. If the j-th barrier is not breached the double-no-touch digital V^j_{DNT} pays off $P^j - P^{j-1}$ and the onion option V^{j-1}_{onion} pays off P^{j-1}. Hence, in this case the right-hand side has the same payoff P^j as the left-hand side. If the underlying price S does not stay within the j-th corridor the right-hand side only has the payoff of the onion option V^{j-1}_{onion} because the double-no-touch option is knocked out. However, this is again equal to the payoff of the left-hand side since the inner corridor j is breached and, therefore, only the outer corridors contribute to the payoff of the option $V^j_{onion}(S,t)$. This shows that Eq. (1) holds for an arbitrary number of corridors.

4 Double-no-Touch Digital

After stripping the onion option into double-no-touch digitals it remains to determine the value $V(S,t;S_{min},S_{max},P)$ of these digitals. Under the usual assumptions of the Black-Scholes framework for FX options (see e.g. [Merton, 1990, Hull, 1997] for a summary) the value $V(S,t;S_{min},S_{max},1)$ of a double-no-touch option satisfies the Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - r_f)S\frac{\partial V}{\partial S} - rV = 0 \tag{3}$$

with the boundary conditions

$$V(S_{min},t) = V(S_{max},t) = 0 \tag{4}$$

for all $t < T$ and the final condition

$$V(S,T) = 1 \tag{5}$$

for all $S \in (S_{min},S_{max})$. In the following, it will be shown that the solution of the boundary problem can be given in terms of a Fourier series. For a general outline of this solution technique see [Wilmott, 1998], Section 6.5.3.

4.1 Transformation of the Final Boundary Value Problem

First, reformulate the Black-Scholes PDE in terms of $x = \ln S/S_{min}$ and use the ansatz $V(x,t) = e^{\alpha x + \beta t}U(x,t)$ with

$$\alpha = \frac{1}{2} - \frac{r - r_f}{\sigma^2}, \tag{6}$$

$$\beta = r + \tfrac{1}{2}\sigma^2\alpha^2 \tag{7}$$

such that Eq. (3) reduces to the heat equation[2]

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial x^2} = 0. \tag{8}$$

with boundary conditions $U(0,t) = 0$, $U(L,t) = 0$ and final condition $U(x,T) = e^{-\alpha x - \beta T}$ for $x \in (0,L)$ where $L = \ln S_{max}/S_{min}$ was introduced.

4.2 Existence, Uniqueness and Regularity of Solutions

From the general theory of linear parabolic final boundary value problems we obtain that the simplified final boundary problem (8) has a unique weak solution

$$U \in L^2((0,T),H^1_0((0,L))) \cap H^1((0,T),H^{-1}((0,L))). \tag{9}$$

Moreover, by continuous imbedding we also have

$$U \in C^0([0,T],L^2((0,L))). \tag{10}$$

We note that on the one hand the final and the boundary data in problem (8) are not compatible. Consequently, when we consider $U(x,t)$ as a continuous function w.r.t. t the trace of $U(x,t)$ w.r.t. x cannot exist. In this sense the regularity statement (10) is optimal. On the other hand, from the general theory of analytic semigroups we obtain that actually $U(x,t)$ is smooth for $t < T$. More precisely we have

$$U \in C^\infty([0,T) \times [0,L]). \tag{11}$$

Consequently, our original final boundary problem (3)–(5) has a unique weak solution

$$V \in L^2((0,T),H^1_0((S_{min},S_{max}))) \cap H^1((0,T),H^{-1}((S_{min},S_{max}))).$$

Moreover, $V(S,t)$ has the following additional regularity:

$$V \in C^0([0,T],L^2((S_{min},S_{max}))). \tag{12}$$

$$V \in C^\infty([0,T);[S_{min},S_{max}]). \tag{13}$$

4.3 Solution via Fourier Series Methods

Expanding $U(x,t)$ in terms of eigenfunctions of the Laplace operator $\partial^2/\partial x^2$ which satisfy the boundary conditions (see e.g. [Evans, 1998]) leads to a Fourier series ansatz $U(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin(\pi n x/L)$. Calculating the Fourier coefficients and reinserting $x = \ln S/S_{min}$ yields [Hui, 1996]

$$V(S,t) = 2\pi \left(\frac{S}{S_{min}}\right)^{\alpha} \sum_{n=1}^{\infty} n \frac{1-(-1)^n e^{-\alpha L}}{n^2\pi^2+\alpha^2 L^2} \times e^{\left[-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2-\beta\right](T-t)} \sin\left(\frac{\pi n}{L}\ln\frac{S}{S_{min}}\right). \tag{14}$$

In Fig. 2 the value of the option is plotted for different times to maturities.

4.4 Convergence Rates

A straightforward calculation yields the following estimate for the remainder terms:

$$\begin{aligned} 2\pi \sum_{n=N+1}^{\infty} & \left| \left(\frac{S}{S_{min}}\right)^{\alpha} n \frac{1-(-1)^n e^{-\alpha L}}{n^2\pi^2+\alpha^2L^2} e^{\left[-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2-\beta\right](T-t)} \sin\left(\frac{\pi n}{L}\ln\frac{S}{S_{min}}\right)\right| \\ &\leq \frac{2}{\pi}\left[(S/S_{min})^{\alpha}+(S/S_{max})^{\alpha}\right] e^{-\beta(T-t)} \sum_{n=N+1}^{\infty} \frac{1}{n} e^{-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2(T-t)} \\ &\leq \frac{2}{\pi}\left[1+(S_{max}/S_{min})^{|\alpha|}\right] e^{-\beta(T-t)} \int_N^{\infty} \frac{1}{n} e^{-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2(T-t)}\,dn \\ &= \frac{1}{\pi}\left[1+e^{|\alpha|L}\right] e^{-\beta(T-t)} \int_{\frac{1}{2}\sigma^2\left(\frac{\pi N}{L}\right)^2(T-t)}^{\infty} \frac{1}{y} e^{-y}\,dy. \end{aligned} \tag{15}$$

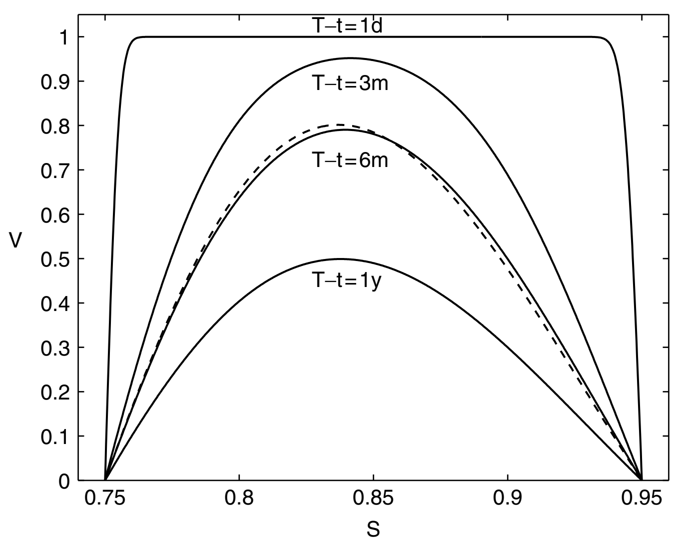

Fig. 2 The value of a double-no-touch option according to Eq. (14) for different times to expiry $T-t$ (full curves). The dashed curve is the contribution of the first term of Eq. (14) to the option value for $T-t=6m$. The figure was generated for the following parameters: $S_{min}=0.75$, $S_{max}=0.95$, $r=5\%$, $r_f=3\%$, $\sigma=10\%$.

TABLE 2
NUMBER OF TERMS NEEDED TO GET A FIVE-DIGIT ACCURACY (ϵ= 0.00001) ON THE RIGHT-HAND SIDE OF EQ. (14) FOR VARIOUS TIMES TO EXPIRY T – t. The TABLE WAS GENERATED FOR THE FOLLOWING PARAMETERS: S_{min} = 0.75, S_{max} = 0.95, r = 5%, r_f = 3%, σ = 10%

T – t	1y	6m	3m	1m	1w	1d
N	4	6	8	14	30	85

This shows that the Fourier series in (14) converges absolutely and uniformly for all values of S for all times $t < T-\varepsilon$ for any $\varepsilon > 0$. However, according to our discussion of the regularity of the solution $V(S,t)$ to the original final boundary value problem (3) the right hand side in (15) diverges logarithmically as t tends to T. It is shown in appendix B that the Fourier series converge uniformly on the entire time interval w.r.t. a different norm. A further simplification of the integral in Eq. (15) can be used to get an explicit expression for the number of terms N that have to be computed in Eq. (14) in order to obtain an approximation of accuracy ε:

$$N = \frac{L}{\pi\sigma}\sqrt{-2\beta - \frac{2}{T-t}\ln\frac{\pi^3\sigma^2(T-t)\,\varepsilon}{2L^2[1+e^{|\alpha|L}]}}. \tag{16}$$

This formula explicitly shows that the number of required terms N will not only increase with decreasing time to expiry $T-t$ and accuracy level ε but also with decreasing volatility σ. However, for parameters typical for the pricing of FX double-no-touch options N seems to be rather small: Using Eq. (16) we calculated the number of terms needed to get an accuracy of 0.00001 for various times to expiry $T-t$ as listed in Table 2. Furthermore, Eq. (16) is a crude lower bound for the number of terms needed to get the desired accuracy. Even with a single term one gets an good approximation of the option value for six month to expiry as can be seen in Fig. 2.

4.5 Gibbs Phenomenon

Although the crude approximation (15) does not give an upper bound for the remainder terms at $t=T$, the Fourier series (14) still converges uniformly to the payoff inside the interval $(S_{min}+\varepsilon, S_{max}-\varepsilon)$ for any $\varepsilon > 0$ as can be anticipated from Fig. 3. However, this figure also shows that close to the boundaries S_{min} and S_{max} the Fourier series overshoots the payoff $V(S,T)=1$ by approximately 18%. Increasing the number of terms does not change the height of the overshoot but only drives the overshooting peaks closer to the edges of the interval (S_{min}, S_{max}); thereby, the series converges to the limit 1 in a growing regime inside the interval. It can be shown (see e.g. [Carslaw, 1930]) that this is a consequence of the

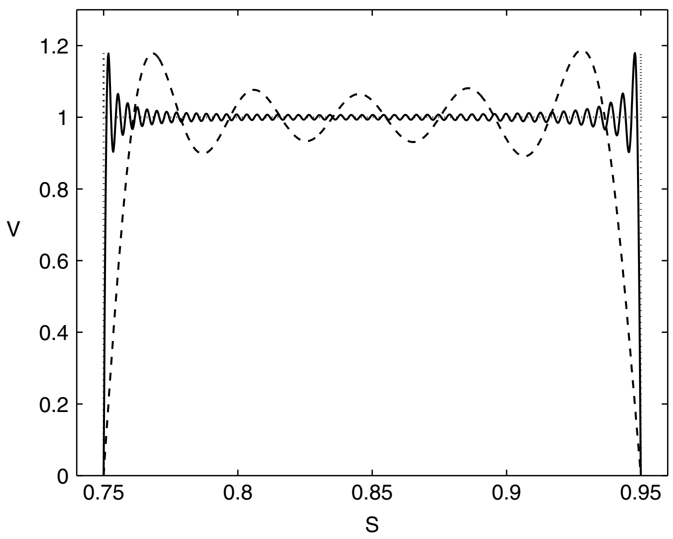

Fig. 3 The dashed and the full curves show the values obtained from the left-hand side of Eq. (14) by truncating the series after 10 and 100 terms, respectively, as a function of the underlying value S. The dotted line shows the graph to which the graphs of the partial sums of the Fourier series will converge as the number of terms tends to infinity. The figure was generated for the following parameters: $S_{min} = 0.75$, $S_{max} = 0.95$, $r = 5\%$, $r_f = 3\%$, $\sigma = 10\%$.

discontinuity at the boundaries and as n becomes infinite the series converges uniformly in the sense of graphs[3] to the dotted curve in Fig. 3 on the entire interval $[S_{min}, S_{max}]$. This is known as the Gibbs phenomenon [Carslaw, 1930][4]. However, for practical purposes the Gibbs phenomenon will not inhibit the use of Eq. (14) because it will only occur at the expiry and for the example discussed in Table 2 even one day before expiry 85 terms are enough to calculate the value of the option with a five-digit accuracy.

4.6 Probability Interpretation

If one inserts the expression for β Eq. (14) can be written as

$$V(S, t) = e^{-r(T-t)} P(S, t), \tag{17}$$

where

$$\begin{aligned} P(S, t) = & 2\pi \left(\frac{S}{S_{min}}\right)^{\alpha} \sum_{n=1}^{\infty} n \frac{1-(-1)^n e^{-\alpha L}}{n^2\pi^2+\alpha^2 L^2} \\ & \times e^{\frac{1}{2}\sigma^2\left[-\left(\frac{\pi n}{L}\right)^2-\alpha^2\right](T-t)} \sin\left(\frac{\pi n}{L}\ln\frac{S}{S_{min}}\right) \end{aligned} \tag{18}$$

is the probability that the underlying value S_t [following the risk neutral process $dS_t = (r - r_f)\,dt + \sigma\, dX_t$] will not leave the corridor (S_{min}, S_{max}) until time T. Eq. (18) yields the same results as formula (2.4) in [Kunitomo and Ikeda, 1992], where this result is generalized to curved boundaries and the probability $P(S, t)$ is given as an infinite sum of cumulative normal distribution functions. The same result can also be obtained by convoluting the payoff with the fundamental solution given in [Cox and Miller, 1965], Eq. (81) with $x = \ln S$.

4.7 The Greeks

In this subsection the greeks Δ, Γ and υ are calculated by differentiating the Fourier series (14). In analogy to the discussion above, we obtain that the differentiated Fourier series converge absolutely and uniformly for $t < T$ and hence represent the respective greeks. However, for $t = T$ the series will always diverge.

4.7.1 Delta

Differentiating Eq. (14) with respect to the underlying S yields:

$$\begin{aligned} \Delta(S, t) = \frac{\partial V}{\partial S} = & \frac{2\pi}{S}\left(\frac{S}{S_{min}}\right)^{\alpha} \sum_{n=1}^{\infty} n \frac{1-(-1)^n e^{-\alpha L}}{n^2\pi^2+\alpha^2 L^2} e^{\left[-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2-\beta\right](T-t)} \\ & \left[\alpha \sin\left(\frac{\pi n}{L}\ln\frac{S}{S_{min}}\right) + \frac{\pi n}{L}\cos\left(\frac{\pi n}{L}\ln\frac{S}{S_{min}}\right)\right]. \end{aligned} \tag{19}$$

In Fig. 4 Δ is plotted as a function of the underlying value S for different times to expiry. The figure demonstrates that Δ can become quite large at the boundaries S_{min} and S_{max} close to the expiry. This indicates that Δ-hedging can be quite involved for these products.

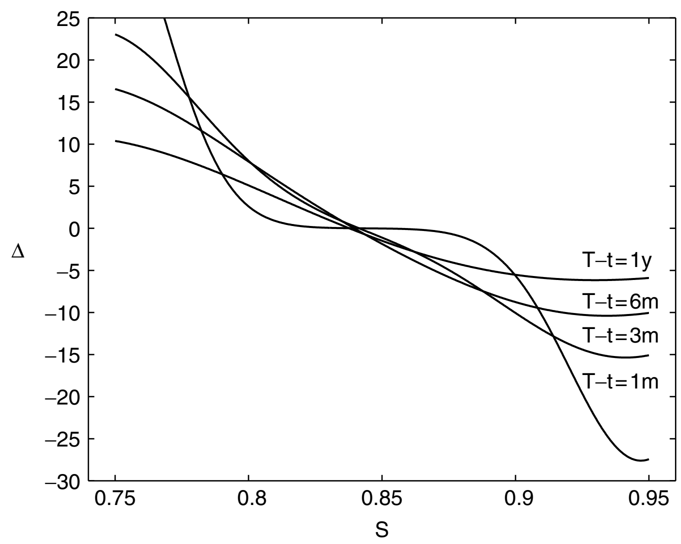

Fig. 4 The delta of the double-no-touch option as a function of the underlying value S according to Eq. (19) for different times to expiry $T - t$. The figure was generated for the following parameters: $S_{min} = 0.75$, $S_{max} = 0.95$, $r = 5\%$, $r_f = 3\%$, %.

4.7.2 Gamma

Differentiating Eq. (19) with respect to the underlying S yields:

$$\begin{aligned}\Gamma(S,t) = \frac{\partial^2 V}{\partial S^2} &= \frac{2\pi}{S^2}\left(\frac{S}{S_{min}}\right)^{\alpha}\sum_{n=1}^{\infty} n\frac{1-(-1)^n e^{-\alpha L}}{n^2\pi^2+\alpha^2L^2} \\ &\times e^{\left[-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2-\beta\right](T-t)}\left\{\left[\alpha^2-\alpha-\left(\frac{\pi n}{L}\right)^2\right]\right. \\ &\times \left.\sin\left(\frac{\pi n}{L}\ln\frac{S}{S_{min}}\right)+\frac{\pi n}{L}(2\alpha-1)\cos\left(\frac{\pi n}{L}\ln\frac{S}{S_{min}}\right)\right\}.\end{aligned} \tag{20}$$

Similar to Δ, Γ can become quite large close to the expiry at the knock-out barriers. Since Γ is the sensitivity against large movements of the asset price this again underlines the hedging difficulties in this regime.

4.7.3 Vega

Differentiating Eq. (14) with respect to the volatility σ yields:[5]

$$\begin{aligned}\upsilon(S,t) = \frac{\partial V}{\partial \sigma} &= 2\pi\left(\frac{S}{S_{min}}\right)^{\alpha}\sum_{n=1}^{\infty}\frac{n}{n^2\pi^2+\alpha^2L^2}e^{\left[-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2-\beta\right](T-t)} \\ &\left\{[1-(-1)^n e^{-\alpha L}]\left[\alpha'(\ln S/S_{min}-L)-\sigma\left(\frac{\pi n}{L}\right)^2(T-t)\right.\right. \\ &\left.\left.-(\sigma\alpha'+\alpha)\alpha\sigma(T-t)-\frac{2\alpha L^2}{n^2\pi^2+\alpha^2L^2}\alpha'\right]+L\alpha'\right\}\sin\left(\frac{\pi n}{L}\ln\frac{S}{S_{min}}\right)\end{aligned} \tag{21}$$

where $\alpha' = \partial\alpha/\partial\sigma = 2(r-r_f)/\sigma^3$. Fig. 6 shows υ as function of the underlying price S for various times to expiry.

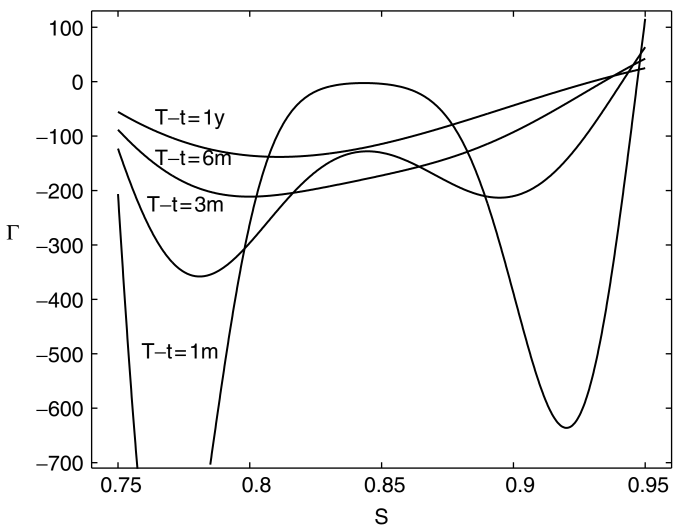

Fig. 5 The gamma of the double-no-touch option as a function of the underlying value S according to Eq. (20) for different times to expiry $T-t$. The figure was generated for the following parameters: $S_{min} = 0.75$, $S_{max} = 0.95$, $r = 5\%$, $r_f = 3\%$, $\sigma = 10\%$.

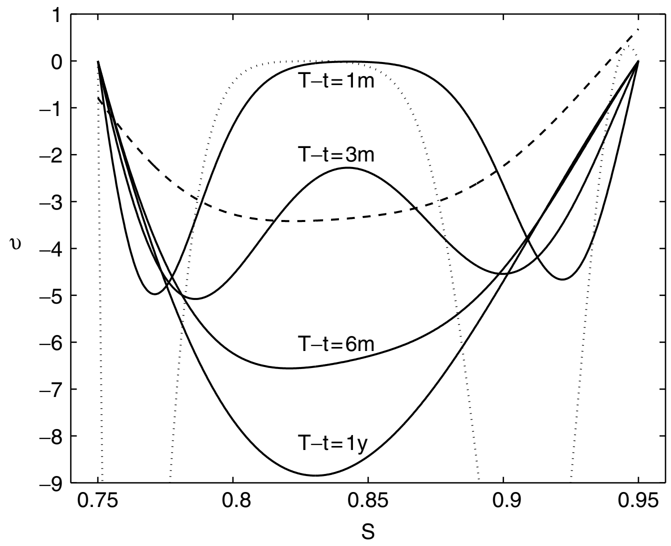

Fig. 6 The vega of the double-no-touch option as a function of the underlying value S according to Eq. (21) for various times to expiry $T-t$ (full lines) and the following parameters: $S_{min} = 0.75$, $S_{max} = 0.95$, $r = 5\%$, $r_f = 3\%$, $\sigma = 10\%$. The dashed curve shows the vega [Eq. (22)] of a digital without barriers for $T-t = 6m$. The dotted curve is the vega according to Eq. (21) for $T-t = 1y$ and the same parameters as above but for $\sigma = 2.5\%$.

The plot suggests that υ is strictly negative for this particular set of parameters. This is somewhat surprising when compared to an option[6] that always pays off 1 monetary unit when the underlying price S at time T is in the interval (S_{min}, S_{max}) regardless of the path that the underlying price takes. The vega of such an option is

$$\begin{aligned}\tilde{\upsilon}(S,t) = e^{-r(T-t)}&\left\{N'(d_2(S_{max}))[\sqrt{T-t}+d_2(S_{max})/\sigma]\right. \\ &\left.-N'(d_2(S_{min}))[\sqrt{T-t}+d_2(S_{min})/\sigma]\right\}\end{aligned} \tag{22}$$

where N' denotes the density of the normal distribution and $d_2(X) = [\ln S/X + (r-r_f-\sigma^2/2)(T-t)]/(\sigma\sqrt{T-t})$. As can be seen from the dashed line in Fig. 6 $\tilde{\upsilon}$ is positive close to the upper boundary S_{max} as long as $r - r_f > 0$. The reason for this is that when the forward price $Se^{(r-r_f)(T-t)}$ of the underlying lies outside the final payoff regime (S_{min}, S_{max}) additional volatility will increase the (risk-neutral) probability for the option to end up in the money and, hence, increase the option's value.

Furthermore, it is surprising that Γ is positive near S_{max} all the same since by taking the derivative with respect to σ of Eq. (3)

$$\frac{\partial\upsilon}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2\upsilon}{\partial S^2}+(r-r_f)S\frac{\partial\upsilon}{\partial S}-r\upsilon = -\sigma S^2\Gamma \tag{23}$$

it can be seen that Γ acts as a source term for vega.

Using a perturbation approach we have shown in appendix A that in contrast to the digital without barriers the digital with barriers has a strictly negative vega in some neighborhood of $r = r_f$ while gamma is positive close to $S_{\max}$ for $r > r_f$ and close to $S_{\min}$ for $r < r_f$. The reason for this peculiar behavior of the vega of the double-no-touch option is that increasing the volatility will increase the probability that the barrier is breached thereby reducing the value of the option. This offsets the positive effect of the increase of the volatility on the expected payoff. However when the dynamics of the underlying asset price is rather drift-dominated ($r - r_f \gg \sigma^2$) this argument is obviously no longer true and υ can become positive for some values of S as is demonstrated by the dotted line in Fig. 6.

5 An Empirical Investigation

In this section, we report some results using *implied volatility*. Although, this concept has been labeled to be 'the wrong number to put in the wrong formula to obtain the right price' [Rebonato, 1999] it is still widely used, especially in risk management and risk controlling. It is also popular among auditors since it complies with IAS (International Accounting Standard) and GAAP. The implied volatilities of the onion option are calculated from their quoted prices by finding the volatility such that Eq. (14) matches the price with all other parameters given. For the set of parameters discussed in the following this is a well-defined procedure since the value is a decreasing function of the volatility. For the examples in table 1 the results are given in table 3. There are some deviations from the implied volatilities taken from at-the-money vanilla options: besides rather obvious sources for these differences such as the non-flat term structure, asynchronous data[7] and day-count conventions the volatility smile can be expected to be a major source for these differences. In [Lipton and McGhee, 2002] more sophisticated pricing techniques focussing on how to deal with the volatility smile are discussed and it is demonstrated that for certain parameters the smile can have a considerable impact on the price of the double-no-touch option. Therefore, the Black-Scholes price can only give a crude indication on what the market price will be.

6 Conclusions

In this article a model-independent decomposition of onion options into double-no-touch options was presented. Thereby, the value of the onion option is the sum of the values of double-no-touch options, whose values can be given in terms of Fourier series. These series converge absolutely and uniformly for all $t < T$ and it is possible to give an explicit formula for the number of terms needed to achieve a desired level of accuracy. This number will diverge as $t \longrightarrow T$ since the Gibbs phenomenon prohibits uniform convergence for $t = T$. However, for parameters most typical for FX options a rather small number of terms is enough to achieve a very good level of accuracy. The Greeks which are also derived from the Fourier series show that hedging will be most involved at the barriers. Furthermore the barriers ensure a peculiar property of vega: There is a range of parameters where vega will be strictly negative for all values of the underlying despite the fact that gamma, which acts as source for vega, is positive in some region. Therefore, the double-no-touch option as well as the onion option can be used by the writer of the option to buy vega.

TABLE 3
IMPLIED VOLATILITY OF THE ONION OPTION LISTED IN TABLE 1 ON THE ISSUE DATE 02/21/2002. FOR THE CALCULATIONS A SPOT VALUE S = 0.8706 AND INTEREST RATES $r = 3.411\%$, $r_f = 2.05\%$ ($T - t = 6m$) [$r = 3.666\%$, $r_f = 2.53\%$ ($T - t = 13m$)] WERE USED.

ISIN	Price 2/21/2002	Implied Volatility	ATM Volatility
DE0006527922	18.30 EUR	8.67 %	9,45 %
DE0006527930	15.20 EUR	8.64 %	9,45 %
DE0006527948	10.05 EUR	8.31 %	9,45 %
DE0006527955	6.70 EUR	7.95 %	9,45 %
DE0006527963	19.20 EUR	8.87 %	9,85 %
DE0006527971	15.60 EUR	8.89 %	9,85 %
DE0006527989	9.65 EUR	8.56 %	9,85 %
DE0006527997	7.15 EUR	8.29 %	9,85 %

A Properties of Vega and Gamma near $r = r_f$

Theorem 1 *Let* $r = r_f$ and $\sigma > 0$. *Then, there exists a neighborhood of* $(r, r_f, \sigma, L, T, t)$ *such that the vega of the double-no-touch option is strictly negative in the interval* (S_{min}, S_{max}).

Proof First, we will prove two properties of vega:

1. For $r = r_f$ υ is strictly negative in the interval (S_{min}, S_{max}).
2. For $r = r_f$ the derivative $\partial\upsilon/\partial S$ with respect to the underlying price S is negative and positive at the lower barrier S_{min} and the upper barrier S_{max}, respectively.

Then, the above theorem follows from the following perturbation argument: Since vega is strictly negative in the interval (S_{min}, S_{max}) a small enough change in the parameters will not make vega positive inside the interval. Since we have $\upsilon(S_{min/max}, t) = 0$ for all values of r and r_f as well as $\partial\upsilon/\partial S(S_{min}, t; r = r_f) > 0$ and $\partial\upsilon/\partial S(S_{max}, t; r = r_f) < 0$ vega can not become positive close to the boundaries $S_{min/max}$ of the interval either.

By comparing Eq. (21) with the time derivative of Eq. (18) we find that

$$\upsilon(S, t)|_{r=r_f} = -e^{-r(T-t)} \frac{2(T-t)}{\sigma} \left.\frac{\partial P}{\partial t}\right|_{r=r_f}. \tag{24}$$

Therefore, we can prove the properties 1 and 2 for $-\partial P/\partial t$ as well as for $\upsilon(S, t)$.

Since P denotes the probability that the underlying price S will remain inside the corridor until the expiry of the option we obviously have $P(S, t) < P(S, t + \delta t)$, which entails $\partial P/\partial t \geq 0$. Furthermore, we know that P fulfils the backward Kolmogorow equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 P}{\partial S^2} + (r - r_f)S\frac{\partial P}{\partial S} = 0 \tag{25}$$

with boundary conditions $P(S_{min/max}) = 0$. Differentiating this with respect to t and using the Feynman-Kac formula yields

$$\frac{\partial P}{\partial t}(S, t) = E\left[\frac{\partial P}{\partial t}(S_{t'\wedge\tau}, t'\wedge\tau)\,\middle|\, S_t = S\right], \tag{26}$$

where S_t follows the risk-neutral geometric Brownian motion and the stopping time τ is defined by $\tau(\omega) = \inf\{u \in [0, t'] | S_u(\omega) \notin (S_{min}, S_{max})\}$. Since $\partial P/\partial t$ is non-negative the expectation of $\partial P/\partial t$ at time t' could only be zero if $\partial P/\partial t(S_{t'}, t') = 0$ for all $S_{t'} \in (S_{min}, S_{max})$. However, we will prove in the following that $\partial P/\partial t$ is positive in some neighborhood of the lower and upper barriers. Therefore, we have

$$\frac{\partial P}{\partial t} > 0 \tag{27}$$

in the interval (S_{min}, S_{max}) for all times $t < T$. This proves property 1.

The derivative of $\partial P/\partial t$ with respect to S is

$$\begin{aligned}\frac{\partial P}{\partial t\partial S}(S, t) = {}& \frac{\pi\sigma^2}{SL}\left(\frac{S}{S_{min}}\right)^{\alpha}\sum_{n=1}^{\infty} n\left[1 - (-1)^n e^{-\alpha L}\right] e^{\frac{1}{2}\sigma^2\left[-\left(\frac{\pi n}{L}\right)^2 - \alpha^2\right](T-t)} \\ & \times\left[\alpha\sin\left(\frac{\pi n}{L}\ln\frac{S}{S_{\min}}\right) + \frac{\pi n}{L}\cos\left(\frac{\pi n}{L}\ln\frac{S}{S_{\min}}\right)\right]\end{aligned} \tag{28}$$

Evaluating this expression at the lower boundary yields

$$\begin{aligned}\frac{\partial P}{\partial t\partial S}(S_{min}, t) = {}& \frac{\pi^2\sigma^2}{S_{min}L^2} e^{-\frac{1}{2}\sigma^2\alpha^2(T-t)}\sum_{n=1}^{\infty} \\ & \times n^2\left[1 - (-1)^n e^{-\alpha L}\right] e^{-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2(T-t)},\end{aligned} \tag{29}$$

which is positive for $\alpha > 0$ because the term $1 - (-1)^n e^{-\alpha L}$ is positive for $\alpha > 0$. At the upper boundary we obtain

$$\begin{aligned}\frac{\partial P}{\partial t\partial S}(S_{max}, t) = {}& \frac{\pi^2\sigma^2}{S_{max}L^2} e^{\alpha L} e^{-\frac{1}{2}\sigma^2\alpha^2(T-t)} \\ & \times\sum_{n=1}^{\infty} n^2\left[(-1)^n - e^{-\alpha L}\right] e^{-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2(T-t)}.\end{aligned} \tag{30}$$

Since we have not restricted our proof of the non-negativity of $\partial P/\partial t$ to any particular set of parameters[8] the following inequality must hold for all values of α:

$$\sum_{n=1}^{\infty} n^2\left[(-1)^n - e^{-\alpha L}\right] e^{-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2(T-t)} \leq 0 \tag{31}$$

Taking the limit $\alpha \to \infty$ yields

$$\sum_{n=1}^{\infty} n^2(-1)^n e^{-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2(T-t)} \leq 0. \tag{32}$$

Using the obvious inequality

$$\sum_{n=1}^{\infty} n^2 e^{-\alpha L} e^{-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2(T-t)} > 0 \tag{33}$$

together with Eq. (31) yields

$$\frac{\partial P}{\partial t\partial S}(S_{max}, t) < 0. \tag{34}$$

Theorem 2 *Let* $\sigma > 0$. *Then,* $\Gamma(S_{min}, t) > 0$ *for* $r < r_f$ *and* $\Gamma(S_{max}, t) > 0$ *for* $r > r_f$ *for all* $t < T$.

Proof At the lower boundary S_{min} the inequality

$$\begin{aligned}\Gamma(S_{min}, t) = {}& \frac{2\pi^2}{S_{min}^2 L}(2\alpha - 1) \\ & \times\sum_{n=1}^{\infty} n^2\frac{1 - (-1)^n e^{-\alpha L}}{n^2\pi^2 + \alpha^2 L^2} e^{\left[-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2 - \beta\right](T-t)} > 0\end{aligned} \tag{35}$$

is obvious for $r < r_f$ $(\alpha > 1/2)$ since this implies $2\alpha - 1 > 0$ and $1 - (-1)^n e^{-\alpha L} > 0$ which ensures that all terms in the sum are positive.

Since $r > r_f$ entails $2\alpha - 1 < 0$ gamma

$$\Gamma(S_{max}, t) = \frac{2\pi^2}{S_{max}^2 L}(2\alpha - 1)\left(\frac{S_{max}}{S_{min}}\right)^{\alpha} e^{-\beta(T-t)} \times \sum_{n=1}^{\infty} n^2 \frac{(-1)^n - e^{-\alpha L}}{n^2\pi^2 + \alpha^2 L^2} e^{-\frac{1}{2}\sigma^2\left(\frac{\pi n}{L}\right)^2 (T-t)} \tag{36}$$

will be positive at the upper boundary for all values of L if

$$\sum_{n=1}^{\infty} n^2 \frac{(-1)^n}{n^2\pi^2 + \alpha^2 L^2} e^{-\frac{1}{2}\sigma^2\left[\left(\frac{\pi n}{L}\right)^2 + \alpha^2\right](T-t)} < 0 \tag{37}$$

holds. First, we note that this inequality is true for $t \ll -1$ since then the odd terms dominate the even terms in the sum. Furthermore, we have in general

$$\frac{\partial}{\partial t} \sum_{n=1}^{\infty} n^2 \frac{(-1)^n}{n^2\pi^2 + \alpha^2 L^2} e^{-\frac{1}{2}\sigma^2\left[\left(\frac{\pi n}{L}\right)^2 + \alpha^2\right](T-t)} = \frac{\sigma^2}{2L^2} \sum_{n=1}^{\infty} n^2 (-1)^n e^{-\frac{1}{2}\sigma^2\left[\left(\frac{\pi n}{L}\right)^2 + \alpha^2\right](T-t)} \leq 0, \tag{38}$$

where the last inequality was taken from Eq. (32). This proves Eq. (37).

B Convergence in L^2

In this appendix we shall derive an estimate for the Fourier series in (14) that holds uniformly on the entire time interval $[0, T]$. However, according to (12) the solution $V(S, t)$ to the original final boundary value problem (3) is only of class $L^2((S_{min}, S_{max}))$ w.r.t. S. For technical reasons it is sensible to consider the equivalent function space $L_w^2((S_{min}, S_{max}))$ where the weight function $w(S)$ is given by

$$w(S) = \frac{S_{min}^{\alpha}}{S^{\alpha+1}}.$$

Again a straightforward calculation yields the following estimate:

$$\begin{aligned}
&\left\{\int_{S_{min}}^{S_{max}} \left[\sum_{n=N+1}^{\infty} A_n e^{-\left[\frac{1}{2}\sigma^2\left(\frac{n\pi}{L}\right)^2 + \beta\right](T-t)} \left(\frac{S}{S_{min}}\right)^{\alpha} \times \sin\left(\frac{n\pi}{L}\ln\frac{S}{S_{min}}\right)\right]^2 w(S)\, dS\right\}^{\frac{1}{2}} \\
&= \left\{\int_0^L \left[\sum_{n=N+1}^{\infty} A_n e^{-\left[\frac{1}{2}\sigma^2\left(\frac{n\pi}{L}\right)^2 + \beta\right](T-t)} \sin\left(\frac{n\pi}{L}x\right)\right]^2 dx\right\}^{\frac{1}{2}} \\
&= \left[\frac{1}{2}\sum_{n=N+1}^{\infty} A_n^2 e^{-\left[\sigma^2\left(\frac{n\pi}{L}\right)^2 + 2\beta\right](T-t)}\right]^{\frac{1}{2}} \\
&\leq \frac{\sqrt{2}P}{\pi}(1 + e^{-\alpha L})\, e^{-\beta(T-t)} \left[\sum_{n=N+1}^{\infty} \frac{1}{n^2} e^{-\sigma^2\left(\frac{n\pi}{L}\right)^2 (T-t)}\right]^{\frac{1}{2}} \\
&\leq \frac{\sqrt{2}P}{\pi}(1 + e^{-\alpha L})\, e^{-\beta(T-t)} \left[\int_N^{\infty} \frac{1}{x^2} e^{-\sigma^2\left(\frac{x\pi}{L}\right)^2 (T-t)}\, dx\right]^{\frac{1}{2}},
\end{aligned}$$

where $A_n = 2\pi n[1 - (-1)^n e^{-\alpha L}]/(n^2\pi^2 + \alpha^2 L^2)$ was introduced for convenience. As a consequence we have the following estimate that is uniform w.r.t. t:

$$\left\{\int_{S_{min}}^{S_{max}} \left[\sum_{n=N+1}^{\infty} A_n e^{-\left[\frac{1}{2}\sigma^2\left(\frac{n\pi}{L}\right)^2 + \beta\right](T-t)} \left(\frac{S}{S_{min}}\right)^{\alpha} \times \sin\left(\frac{n\pi}{L}\ln\frac{S}{S_{min}}\right)\right]^2 w(S)\, dS\right\}^{\frac{1}{2}} \leq \frac{\sqrt{2}P}{\pi}(1 + e^{-\alpha L})\frac{1}{\sqrt{N}}. \tag{40}$$

In particular, the Fourier series in (14) converges in the function space $C^0([0, T], L^2((S_{min}, S_{max})))$ with rate $1/\sqrt{N}$.

FOOTNOTES & REFERENCES

1. This means $S^2_{min} < S^3_{min} < S^3_{max} < S^2_{max}$.
2. In contrast to the heat equation from physics which is forward parabolic, Eq. (8) is backward parabolic because calendar time t has not been substituted by time to maturity $\tau = T - t$.
3. Therefore, a norm has to be introduced that measures the distance between two curves as subsets of R^2.
4. This was already mentioned in the financial context in [Wilmott, 1998].
5. Note that α and β also depend on σ.
6. The value of such an option is the same as of a portfolio containing a long position in a binary call with strike S_{min} and a short position in a binary call with strike S_{max}.
7. The market data and the quotes by Commerzbank have been taken on the same day but not at the time.
8. Our proof for the negativity of vega is of course restricted to a neighbourhood of $r = r_f$ since Eq. (24) holds only there.

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