Option Pricing with Finite Elements

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Abstract: The Finite Element (FE) Method is a standard numerical technique in engineering and natural sciences. Some authors have successfully employed FE to option pricing problems. This paper delivers a gentle and non-technical introduction to basic ideas and concepts of this method.

1 Introduction

Finite Differences (FD) were already in use by Newtown, Leibnitz and other early developers of calculus. A paper by Courant [Courant, 1943] is generally considered the starting point of Finite Elements (FEs), although some building blocks of the FE method can be traced back much earlier as to the work of Schellbach, Ritz, Galerkin and some others. The Courant paper had little influence at that time since computers were not readily available. This changed in the 1950's and structural engineers soon realized to use the new computing power for their discretization methods in which a structure is envisaged as divided into elements with *locally* defined stresses and strains. Pioneering works of this age are [Turner et al., 1956] and [Argyris, 1955]. The expression *Finite Elements* has been coined by [Clough, 1960]. In 1965 it was realized, that FE could be employed to all field problems that could be formulated as variational problems [Zienkiewicz and Cheung, 1965]. From then on FE conquered many other fields of natural science and engineering other than structural analysis. In parallel, the mathematical foundations were developed including proofs for error bounds, convergence, and stability. Nowadays, there are FE approaches for virtually any mathematical or physical problem that can be described with equations of calculus, i.e. differential, integral, integrodifferential, and variational equations.

In the late 1990's the first applications of FEs to option pricing problems were delivered by the two PhD theses [Tomas, 1996] and [Jackson, 1999]. Both publications do not solve directly the Black-Scholes PDE but a transformation. This way, the applicability to real-world problems is somewhat reduced since discrete dividends, discrete fixings, etc. cannot usually be integrated into the transformed pricing equation. In the following, several papers were published using FE for various pricing problems: Term structure models [Topper, 1998], passport options [Topper, 2001a], reverse convertibles [Topper, 2001b], convertibles ([Barone-Adesi et al., 2003], [Ouachani04 and Zhang, 2004]), and various exotic options ([Jackson and süli, 1997], [Topper, 2000], [Pooley et al., 2000] [Forsyth et al., 1999]). The reader interested in *how* FEs work is referred to [Topper, 2005]; the reader interested in *why* these methods work, should look up [Alberty, 2004].

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Before going into the details of the FE method the author wants to outline succinctly its advantages:

- A solution for the entire domain is computed, instead of isolated nodes as in the case of FD.
- The boundary conditions involving derivatives are difficult to handle with FD. Neumann conditions, however, are often easier to obtain than Dirichlet conditions when estimating the behaviour of the option as the price of the underlying goes to infinity. FE techniques easily can incorporate boundary conditions involving derivatives.
- In addition, FE can easily deal with high curvature. In most FE codes this is achieved by adaptive remeshing, a technique well-developed in theory and in practice.
- The irregular shapes of the PDE's domain can easily be handled, while in a FD setting the placing of the gridpoints is difficult. These irregular domains arise naturally when knock-out barriers are imposed on a multiple-asset option. Irregular shapes can also arise when only parts of the PDE's domain are to be approximated numerically because some parts can be determined by financial reasoning. Irregular domains also arise in the pricing of convertible bonds [Ouachani and Zhang, 2004].
- Most academic papers are concerned with pricing only while the majority of practioners are at least as much interested in measures of sensitivity to those prices. Some of these measures of sensitivity, commonly called Greeks, can be obtained more exactly with FE.
- Many FE codes (such as PDE2D, used for this paper) allow local refinement. This allows precise local information without having to solve the problem with higher accuracy on the entire domain. PDE2D also employs adaptive remeshing. This feature automatically leads to local refinement in areas of high curvature, for example near to the strike price or close to the barrier.
- FE can easily be combined with infinite elements or boundary elements for the treatment of (semi-)infinite domains. This is common practice in engineering while in finance usually artificial boundary conditions are introduced.

These advantages come at the cost of a more complicated method compared to FD. In the next section a brief but also easy-to-read presentation of the FE method will be delivered. Some examples demonstrate the usefulness of FE for financial problems.

2 The Method of Finite Elements

Problems arising in option pricing are usually of the form

$$u_t = L[u] - f \tag{1}$$

with $L[u]$ being a differential operator of second order. Following the engineering terminology we call such problems *dynamic* in contrast to problems such as $L(u) - f = 0$ which are labelled *static*. Most pricing problems are dynamic with the exception of some perpetual options, which can be priced using static models [Lipton, 2001]. Static problems also arise in the context of computing steady-state distributions and first exit times. We follow the common practice to discretize the spatial variables with FE and time (to maturity) with FD, i.e. at this point we will not discuss space-time-elements, which -at least to the knowledge of the author- have not been applied to option pricing problems in the public domain literature. All FE methods in use today belong to the *Method of Weighted Residuals* (MWR). This method will be explained with the help of a static problem in one variable x.

The first step of a FE approach is to subdivide the domain Ω into non-overlapping sub-domains. Besides, the finite elements need to cover the domain completely. In 1D problems this is reduced to divide a straight line into intervals as in fig. 1.

The approximate solution $\tilde{u}$ (also called *interpolation function*) is to be of the following form:

$$\tilde{u} = \sum_{i=1}^{N+1} \tilde{u}_i \phi_i \tag{2}$$

The ϕ_i are called *shape functions*, *basis functions*, or *trial functions*. The choice of appropriate shape functions depends on various criteria, such as the order of the differential equation at hand and the order of the derivative of the approximate solution needed. The weights $\tilde{u}_i$ are to be determined by the numerical algorithm in a way such that $u \approx \tilde{u}$. This is made more precise by introducing the residual $R = L(\tilde{u}) - f$, which is to be minimized. This can be accomplished in various fashions. Since there are $N+1$ unknowns, also $N+1$ equations need to be established to determine them:

$$\int_\Omega R W_j \, dx \stackrel{!}{=} 0 \qquad \forall \; j = 1, \ldots, N+1 \tag{3}$$

Inserting the definition of the residual into eq. (3):

$$\int_\Omega (L(\tilde{u}) - f) W_j \, dx = 0 \qquad \forall \; j = 1, \ldots, N+1 \tag{4}$$

$$\Leftrightarrow \qquad \int_\Omega L(\tilde{u}) W_j \, dx = \int_\Omega f \, W_j \, dx \qquad \forall \; j = 1, \ldots, N+1 \tag{5}$$

For linear operators, $L(\cdot)$ on the left side of eq. (5) simplifies to:

$$\int_\Omega L\left(\sum_i \tilde{u}_i \phi_i\right) W_j \, dx = \int_\Omega \sum_i \tilde{u}_i L(\phi_i) \; W_j \, dx = \sum_i \tilde{u}_i \underbrace{\int_\Omega L(\phi_i) \; W_j \, dx}_{=K_{ij}} \tag{6}$$

Figure 1: Discretization of a 1D problem

The right side of eq. (5) can be rewritten as:

$$\mathbf{F} = \begin{pmatrix} \int_\Omega f W_1 \, dx \\ \vdots \\ \int_\Omega f W_N \, dx \\ \int_\Omega f W_{N+1} \, dx \end{pmatrix} \tag{7}$$

so that the FE solution of a *linear* differential equation reduces to solving a system of *linear* equations given by:

$$\mathbf{K}\tilde{\mathbf{u}} = \mathbf{F} \tag{8}$$

The discussion is not complete at this point. Nothing has been said about the BC so far. Also, the weighting functions need to be specified. Popular choices are:

- The *Galerkin Method*: The weighting function is given by $W_i(x) = \phi_i(x)$ so that:

$$\int_\Omega R(x, \tilde{\mathbf{u}})\phi_i(x)dx \stackrel{!}{=} 0 \quad \forall i \tag{9}$$

i.e. the appropriate weighting function is the shape function ϕ_i. The Galerkin method is very popular in engineering because it produces the same solution as the Ritz variational method when using the same trial function. This is of little relevance in financial applications since hardly any problem is self-adjoint so that an equivalent formulation as a problem from the realms of calculus of variations is not possible. However, most applications of FE in finance have so far applied the Galerkin criterion. The most striking advantage of the Galerkin method is that it produces nicely structured matrices, which can quickly be solved with appropriate solvers. The Galerkin approach also has a distinct disadvantage. It can be applied only to problems which can be stated in divergence form, i.e.

$$L[u] = \sum_{i,k=1}^{n} (a_{ik}u_i)_k + \sum_{i=1}^{n} b_i u_i + cu \tag{10}$$

Not all pricing problems can be cast into this format, such as the Hamilton-Jacobi-Bellmann equations arising in the pricing of passport options or optimal portfolios.

- The *Collocation Method*: $W_i(x) = \delta(x - x_i)$ (with δ being the Dirac delta function) so that

$$\int_\Omega R(x, \tilde{\mathbf{u}}) \; \delta(x - x_i) \; dx \stackrel{!}{=} R(x_i; \tilde{\mathbf{u}}) \quad \forall i \tag{11}$$

In principle, the collocation points can be positioned anywhere in the domain (including the boundary) not necessarily following a particular pattern. In practice, the collocation method is usually used in combination with Hermite basis functions which make the Gauss integration points the best choice for the collocation points.

- The *Least Square Method*:

$$\frac{\partial}{\partial \tilde{u}_i} \int_\Omega [R(x, \tilde{\mathbf{u}})]^2 \; dx \stackrel{!}{=} 0 \quad \forall i \tag{12}$$

Because of Leibnitz's rule, this is equivalent to:

$$\int_\Omega R(x, \tilde{\mathbf{u}}) \; \underbrace{\frac{\partial}{\partial \tilde{u}_i} R(x, \tilde{\mathbf{u}})}_{=W_i(x)} \; dx \stackrel{!}{=} 0 \quad \forall i \tag{13}$$

The Least Square FE method offers some advantages when dealing with convection-dominated problems. For details see ([Jiang, 1998]; [Topper, 2005], sec. 4.6.4).

What is the reason of choosing shape function with local support? The MWR can be employed to shape functions with global support as well. However, the solution of the resulting systems of equations becomes increasingly difficult to solve. In the linear case the system (8) becomes ill-conditioned as the number of parameters of a global shape function increases. This problem is circumvented by employing shape functions restricted to finite elements of the (spatial) domain.

Integrating time into this framework is established via semidiscretization, i.e. employing an interpolation function of the following form:

$$\tilde{u}(x, t) = \sum_{i=1}^{N+1} \tilde{u}_i(t)\phi_i(x) \tag{14}$$

Semidiscretization can be applied to any member of the MWR family. This will be demonstrated by employing it to a parabolic differential equation (including the Black-Scholes model and the Vasicek model as special types):

$$\dot{u} = a_0 u'' + a_1 u' + a_2 u + f \tag{15}$$

We assume that both boundary conditions are given as mixed conditions at $x^{\min}$ and $x^{\max}$.

$$\alpha_1 u(x^{\min}) + \beta_1 u'(x^{\min}) = \gamma_1 \tag{16}$$

$$\alpha_2 u(x^{\max}) + \beta_2 u'(x^{\max}) = \gamma_2 \tag{17}$$

To render the problem well-posed an IC is needed as well:

$$u(t_0, x) = u_0(x) \tag{18}$$

First we develop a Galerkin FE model with a linear shape function:

$$\tilde{u}(x, t) = \sum_{i=1}^{N+1} \tilde{u}_i(t) n_i(x) \tag{19}$$

with:

$$n_i(x_j) = 1 \quad \forall \quad i = j \tag{20}$$

$$n_i(x_j) = 0 \quad \forall \quad i \neq j \tag{21}$$

The function n_i has only local support. It equals one at node i and vanishes at every other node. The residual is defined as above:

$$R(x, t, \tilde{u}_1, \ldots, \tilde{u}_{N+1}) = (a_0\tilde{u}')' + (a_1 - a_0')\tilde{u}' + a_2\tilde{u} + f - \tilde{u}_t \tag{22}$$

The Galerkin criterion requires:

$$\int_{x^{\min}}^{x^{\max}} R\, n_k\, dx \stackrel{!}{=} 0 \quad \forall\, k = 1, \ldots, N+1 \tag{23}$$

$$\Leftrightarrow \int_{x^{\min}}^{x^{\max}} \left[(a_0\tilde{u}')' + (a_1 - a_0')\tilde{u}' + a_2\tilde{u} + f - \tilde{u}_t\right] n_k\, dx \stackrel{!}{=} 0 \quad \forall\, k = 1, \ldots, N+1 \tag{24}$$

Integrating the first term to eliminate the second order derivative:

$$\int_{x^{\min}}^{x^{\max}} (a_0\tilde{u}')'\, n_k\, dx = \left[a_0\tilde{u}' n_k\right]_{x^{\min}}^{x^{\max}} - \int_{x^{\min}}^{x^{\max}} a_0\tilde{u}'\, n_k'\, dx \tag{25}$$

$$= \left[a_0\tilde{u}'(x^{\max})n_k(x^{\max})\right] - \left[a_0\tilde{u}'(x^{\min})n_k(x^{\min})\right] - \int_{x^{\min}}^{x^{\max}} a_0\tilde{u}'\, n_k'\, dx$$

$$= a_0 n_k(x^{\max})\left[\frac{\gamma_2 - \alpha_2 u(x^{\max})}{\beta_2}\right] - a_0 n_k(x^{\min})\left[\frac{\gamma_1 - \alpha_1 u(x^{\min})}{\beta_1}\right] - \int_{x^{\min}}^{x^{\max}} a_0\tilde{u}'\, n_k'\, dx$$

Inserting the above result in eq. (24) leads to:

$$\int_{x^{\min}}^{x^{\max}} \left[a_0\tilde{u}' n_k' - (a_1 - a_0')\tilde{u}' n_k - a_2\tilde{u} n_k - f n_k + n_k\dot{\tilde{u}}\right] n_k\, dx$$

$$\stackrel{!}{=} a_0 n_k(x^{\max})\left[\frac{\gamma_2 - \alpha_2 u(x^{\max})}{\beta_2}\right] - a_0 n_k(x^{\min})\left[\frac{\gamma_1 - \alpha_1 u(x^{\min})}{\beta_1}\right] \quad \forall\, k = 1, \ldots, N+1 \tag{26}$$

Next we consider only the LHS. Inserting eq. (19) into eq. (26) results in:

$$\int_{x^{\min}}^{x^{\max}} \left[a_0 n_k'\left(\sum_{i=1}^{N+1}\tilde{u}_i n_i'\right) - (a_1 - a_0')n_k\left(\sum_{i=1}^{N+1}\tilde{u}_i n_i'\right) - a_2 n_k\left(\sum_{i=1}^{N+1}\tilde{u}_i n_i\right) - f n_k + n_k\left(\sum_{i=1}^{N+1}\dot{\tilde{u}}_i n_i\right)\right] dx$$

$$= \sum_{i=1}^{N+1}\left[\int_{x^{\min}}^{x^{\max}} \left(a_0 n_k' n_i' - (a_1 - a_0')n_k n_i' - a_2 n_k n_i\right) dx\right]\tilde{u}_i + \sum_{i=1}^{N+1}\left[\int_{x^{\min}}^{x^{\max}} (n_k n_i)\, dx\right]\dot{\tilde{u}}_i - \int_{x^{\min}}^{x^{\max}} n_k f\, dx$$

Inserting the above result into eq. (26) leads to:

$$\sum_{i=1}^{N+1}\left[\int_{x^{\min}}^{x^{\max}} \left(a_0 n_k' n_i' - (a_1 - a_0')n_k n_i' - a_2 n_k n_i\right) dx\right]\tilde{u}_i + \delta_{kN+1}\frac{\alpha_2}{\beta_2}\tilde{u}_{N+1} - \delta_{k1}\frac{\alpha_1}{\beta_1}\tilde{u}_1 + \sum_{i=1}^{N+1}\left[\int_{x^{\min}}^{x^{\max}} (n_k n_i)\, dx\right]\dot{\tilde{u}}_i = \int_{x^{\min}}^{x^{\max}} n_k f\, dx + \delta_{kN+1}\frac{\gamma_2}{\beta_2} - \delta_{k1}\frac{\gamma_1}{\beta_1}\tilde{u}_1 \tag{27}$$

$$\forall\, k = 1, \ldots, N+1$$

$$\Leftrightarrow \mathbf{A}\tilde{\mathbf{u}} + \mathbf{B}\dot{\tilde{\mathbf{u}}} = \mathbf{q} \tag{28}$$

In order to solve the system of initial value problems (28), a starting value $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_0(0)$ for each ODE is needed. These starting values are obtained by discretizing the initial condition eq. (18) associated with the PDE under inspection.

This approach cannot deal with Dirichlet conditions, i.e. $\beta_1 = 0$ or $\beta_2 = 0$. The Dirichlet BC is enforced with a constraint. Let us consider the case of one Dirichlet condition which is located at $x^{\min}$. Eq. (28) written out in full reads:

$$a_{11}\tilde{u}_1 + a_{12}\tilde{u}_2 + a_{13}\tilde{u}_3 + \cdots + b_{11}\dot{\tilde{u}}_1 + b_{12}\dot{\tilde{u}}_2 + b_{13}\dot{\tilde{u}}_3 + \cdots = q_1(t) \tag{29}$$

$$a_{21}\tilde{u}_1 + a_{22}\tilde{u}_2 + a_{23}\tilde{u}_3 + \cdots + b_{21}\dot{\tilde{u}}_1 + b_{22}\dot{\tilde{u}}_2 + b_{23}\dot{\tilde{u}}_3 + \cdots = q_2(t) \tag{30}$$

$$a_{31}\tilde{u}_1 + a_{32}\tilde{u}_2 + a_{33}\tilde{u}_3 + \cdots + b_{31}\dot{\tilde{u}}_1 + b_{32}\dot{\tilde{u}}_2 + b_{33}\dot{\tilde{u}}_3 + \cdots = q_3(t) \tag{31}$$

$$\vdots$$

The first equation of the above system is replaced by the Dirichlet BC:

$$\alpha_1\tilde{u}_1 = \gamma_1(t) \tag{32}$$

$$a_{21}\tilde{u}_1 + a_{22}\tilde{u}_2 + a_{23}\tilde{u}_3 + \cdots + b_{21}\dot{\tilde{u}}_1 + b_{22}\dot{\tilde{u}}_2 + b_{23}\dot{\tilde{u}}_3 + \cdots = q_2(t) \tag{33}$$

$$a_{31}\tilde{u}_1 + a_{32}\tilde{u}_2 + a_{33}\tilde{u}_3 + \cdots + b_{31}\dot{\tilde{u}}_1 + b_{32}\dot{\tilde{u}}_2 + b_{33}\dot{\tilde{u}}_3 + \cdots = q_3(t) \tag{34}$$

$$\vdots$$

In short, the constrained set of equations can be written as:

$$\mathbf{M}\dot{\tilde{\mathbf{u}}} + \mathbf{K}\tilde{\mathbf{u}} = \mathbf{f} \tag{35}$$

The numerical solution of the above system of ordinary initial value problems is usually achieved with FE. Note, that systems resulting from FE dicretizations tend to be stiff.

As a second example, the collocation FE method without specifying a shape function is discussed. The collocation method requires that the interpolation function and its derivatives replace the function u to be approximated. Inserting the derivatives of the basis function (14) into the parabolic PDE (15) without Neumann conditions leads to the following system of ODEs:

$$\sum_{i=1}^{N+1}\dot{\tilde{u}}_i(t)\phi_i(x_j) = a_0\sum_{i=1}^{N+1}\tilde{u}_i(t)\phi_i''(x_j) + a_1\sum_{i=1}^{N+1}\tilde{u}_i(t)\phi_i'(x_j) + a_2\sum_{i=1}^{N+1}\tilde{u}_i(t)\phi_i(x_j) + f \tag{36}$$

$$= \sum_{i=1}^{N+1}[a_0\phi_i''(x_j) + a_1\phi_i'(x_j) + a_2\phi_i(x_j)]\tilde{u}_i(t) + f \tag{37}$$

for $j = 1, \ldots, N+1$. In order to change this expression to matrix notation, we define the following $(N+1) \times (N+1)$ matrices:

$$\mathbf{M} = [\phi_i(x_j)]_{(ji)} \tag{38}$$

$$\mathbf{N} = [a_0\phi_i''(x_j) + a_1\phi_i'(x_j) + a_2\phi_i(x_j)]_{(ji)} \tag{39}$$

Eq. (36) can then be rewritten as:

$$\mathbf{M}\dot{\tilde{\mathbf{u}}} = \mathbf{N}\tilde{\mathbf{u}} + \mathbf{f} \tag{40}$$

Note, that the above system consists of initial value ODEs just as the semidiscretization of a parabolic PDE with the Galerkin method; compare system (35). The integration of the Dirichlet BC is the same as in the Galerkin method. The first line and last line of the system (40) is replaced by the BC as in eq. (29) to (34).

3 Examples

3.1 A European Put

All examples in this paper have been computed with PDE2D, a FORTRAN-based PDE solver developed by Granville Sewell and described in [Sewell, 2000].

As a first example the author presents the Black-Scholes model for plain vanilla European put with the market data given in table 1. The numerical values in the first scenario have been achieved with 100 elements of equal length using cubic Hermite basis functions and a collocation FE approach. Also, 100 time steps of equal length are chosen employing the backward Euler method. The second scenario uses 200 elements of the same type and an adaptive backward Euler scheme using 163 time-steps. The following boundary conditions are employed:

$$V(0) = e^{-r(T-t)}E \tag{41}$$

$$V(100) = 0 \tag{42}$$

In the second scenario, the discretization leads to an approximation error beyond the fourth decimal digit as reported in table 2.

TABLE 1: DATA FOR A PLAIN VANILLA PUT

Parameter	Symbol	Value
Strike price	E	40
Interest rate	r	0.1
Volatility	σ	0.2
Maturity	T	0.5 year

3.2 An American Put

There are various techniques to integrate early exercise into the Black-Scholes model. One especially suited method within the FE framework is to augment the Black-Scholes PDE by a penalty function [Zvan et al., 1998]:

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} \\ - rV + \left(c_{penalty}\ \{\min\left[V - \max(E-S, 0), 0\right]\}^2\right) = 0\end{aligned} \tag{43}$$

Also one of the two boundary conditions has to be changed to $V(0) = E$. The constant $c_{penalty}$ is set to $c_{penalty} = 10,000$. The discretization is the same as in the scenarios from sec. 3.1. The nonlinear system to be solved at each time-step resulting from the nonlinear PDE (43) is solved with a variant of the Newton method. Using the same market data as above we achieve the results reported in table 3. The adaptive time-steps in the second scenario consumes 166 time-steps. Since no analytical solution is available, a trinomial tree (with 100 steps) as implemented by [Haug, 1997] is used as a benchmark.

A Basket Option

No closed-form solution for the option on a portfolio of assets is known in the Black-Scholes setting. As an example a two-asset basket put option with market data as given in table 4 is computed. The FE computations are compared to Monte Carlo results which were achieved with 100,000 simulations employing a simple antithetic technique. The FE discretization employs are regular triangularization as plotted in fig. 2. A Galerkin approach with elements of degree 3 are used. The contour plot fig. 3

TABLE 2: RESULTS FOR A PLAIN VANILLA PUT

Underlying	Analytical	Analytical	Scenario 1		Scenario 2	
	Premium	Δ	FE-Premium	FE-Δ	FE-Premium	FE-Δ
35	3.9318	–0.6985	3.9334	–0.6994	3.9318	–0.6985
37	2.6811	–0.5505	2.6806	–0.5515	2.6811	–0.5505
39	1.7292	–0.4031	1.7273	–0.4036	1.7292	–0.4031
41	1.0557	–0.2746	1.0533	–0.2746	1.0557	–0.2746
43	0.6114	–0.1747	0.6094	–0.1743	0.6114	–0.1747
45	0.3370	–0.1044	0.3358	–0.1039	0.3370	–0.1044
47	0.1774	–0.0588	0.1770	–0.0585	0.1774	–0.0588

TABLE 3: RESULTS FOR A AMERICAN PUT

Under-lying	Trinomial Premium	Trinomial Δ	Scenario 1 FE-Premium	Scenario 1 FE-Δ	Scenario 2 FE-Premium	Scenario 2 FE-Δ
35	5.0000	–1.0000	4.9939	–0.9987	4.9941	–0.9970
37	3.2453	–0.7411	3.2441	–0.7360	3.2461	–0.7343
39	2.0209	–0.5054	2.0154	–0.5026	2.0195	–0.5022
41	1.2008	–0.3186	1.1959	–0.3259	1.2000	–0.3262
43	0.6822	–0.2026	0.6778	–0.2001	0.6807	–0.2006
45	0.3692	–0.1245	0.3675	–0.1164	0.3692	–0.1196
47	0.1914	–0.0613	0.1913	–0.0643	0.1919	–0.0647

TABLE 4: DATA FOR A CALL ON A BASKET

Parameter	Value
First asset price	18
Weight first asset	1
Second asset price	20
Weight second asset	1
Strike price	40
Interest rate	0.1
Dividend Yields	0.0
Correlation	0.5

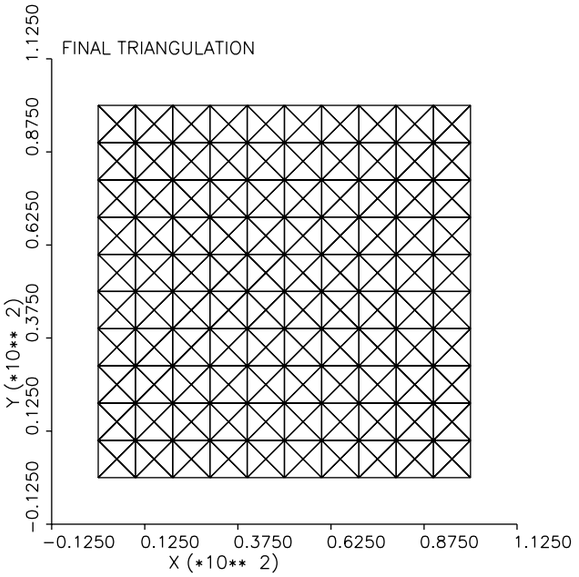

Figure 2: Triangulation of spatial domain of a basket option

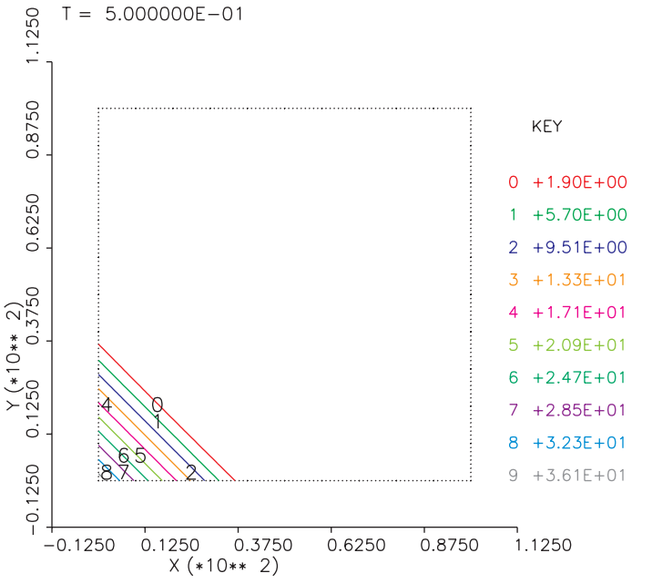

Figure 3: Contour plot of the spatial domain of a basket option

TABLE 5: RESULTS PUT OPTION ON A BASKET

Volatility σ_1^2	Volatility σ_2^2	Time to Maturity 0.05	0.5	0.95	
		1.8044	0.9599	0.6015	MC
	0.1	1.8065	0.9543	0.6043	FEM
		1.8354	1.4825	1.2453	MC
0.1	0.2	1.8341	1.4764	1.2405	FEM
		1.9109	2.0087	1.9225	MC
	0.3	1.9138	2.0187	1.9270	FEM
		1.8271	1.4120	1.1607	MC
	0.1	1.8275	1.4127	1.1601	FEM
		1.8859	1.8835	1.7758	MC
0.2	0.2	1.8856	1.8833	1.7754	FEM
		1.9816	2.3941	2.4389	MC
	0.3	1.9830	2.3942	2.4389	FEM
		1.8906	1.8941	1.7649	MC
	0.1	1.8915	1.8948	1.7647	FEM
		1.9683	2.3301	2.3557	MC
0.3	0.2	1.9687	2.3298	2.3555	FEM
		2.0739	2.8112	2.9985	MC
	0.3	2.0747	2.8119	2.9979	FEM

results from triangles of degree 3 and $\sigma_1 = \sigma_2 = 0.1$. Time integration is achieved with a backward Euler scheme using 200 time steps. The following boundary conditions are chosen:

$$V(0, S_2, \tau) = \text{BS-price of put on } S_2 \quad (44)$$

$$V(S_1, 0, \tau) = \text{BS-price of put on } S_1 \quad (45)$$

$$V(100, S_2, 0) = 0 \quad (46)$$

$$V(S_1, 100, 0) = 0 \quad (47)$$

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